

# BLOCH'S CONJECTURE FOR GENERALIZED BURNIAT TYPE SURFACES WITH $p_g = 0$

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**ABSTRACT.** The aim of this article is to prove Bloch's conjecture, asserting that the group of rational equivalence classes of zero cycles of degree 0 is trivial for surfaces with geometric genus zero, for regular generalized Burniat type surfaces. The technique is the method of "enough automorphisms" introduced by Inose-Mizukami in a simplified version due to the first author.

## INTRODUCTION

Let  $S$  be a smooth projective surface and let

$$A_0(S) = \bigoplus_{i=-\infty}^{\infty} A_0^i(S)$$

be the group of rational equivalence classes of zero cycles on  $S$ . Then *Bloch's conjecture* asserts the following:

**Conjecture** ([Blo75]). *Let  $S$  be a smooth surface with  $p_g(S) = 0$ . Then the kernel  $T(S)$  of the natural morphism:*

$$A_0^0(S) \longrightarrow \text{Alb}(S)$$

*is trivial.*

The conjecture has been proven for surfaces  $S$  with Kodaira dimension  $\text{kod}(S) \leq 1$  by Bloch, Kas and Lieberman (cf. [BKL76]) and has been verified for several examples, see e.g. [Bar85], [Bau14], [CC13], [IM79], [Keu88], [Voi92]. Thanks to a result of S. Kimura (cf. [Kim05]), all product quotient surfaces (i.e. minimal models of  $(C_1 \times C_2)/G$ , where  $G$  is a finite group acting on the product of two curves of genus at least 2) with  $p_g = 0$  satisfy Bloch's conjecture (cf. [BCGP12]).

*Burniat surfaces* are surfaces of general type with invariants  $p_g = 0$  and  $K^2 = 6, 5, 4, 3, 2$  whose birational models were constructed by P. Burniat in 1966 (cf. [Bur66]) as singular bidouble covers of the projective plane. In 1994, M. Inoue (cf. [Ino94]) reconstructed them as quotients of a divisor in a product of three elliptic curves by a finite group acting freely (see also [BC12] and [BC13]).

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Following and generalizing Inoue’s approach, in the recent paper [BCF], we construct and classify a new class of surfaces of general type “*generalized Burniat type surfaces*”. These surfaces have invariants  $K^2 = 6$  and  $0 \leq p_g = q \leq 3$  and have been constructed as quotient of a divisor of multi-degree  $(2, 2, 2)$  in a product of three elliptic curves by a free  $(\mathbb{Z}/2\mathbb{Z})^3$ -action (see Section 3). Generalized Burniat type surfaces form 16 irreducible families; four families have  $p_g = 0$  and form four connected components  $\mathfrak{N}_i$  of the Gieseker moduli space  $\mathfrak{M}_{1,6}^{can}$ . Each component is irreducible, generically smooth, normal and unirational of dimension 4 in two cases and 3 in the others.

The main result of this note is to show that Bloch’s conjecture holds for generalized Burniat type surfaces with  $p_g = 0$ . The proof uses the method of “enough automorphisms” introduced by Inose and Mizukami (cf. [IM79]) and refined by Barlow (cf. [Bar85]), but in a simpler way (cf. [Bau14]).

**Theorem.** *Let  $S$  be a generalized Burniat type surface with  $p_g(S) = 0$ . Then  $S$  verifies Bloch’s conjecture:  $T(S) = A_0^0(S) = 0$ .*

In [BCF] the authors show among others the following result:

**Theorem.** *Let  $S$  be any surface whose moduli point lies in the connected component of the Gieseker moduli space of surfaces of general type of a regular generalized Burniat type surface, then  $S$  is a generalized Burniat type surface.*

More precisely, in [BCF] it is proven that generalized Burniat type surfaces form exactly four irreducible connected components in their moduli space (cf. Theorem 4.6). Therefore our result proves Bloch’s conjecture for each surface in the connected component of any regular generalized Burniat type surface.

## 1. BLOCH’S CONJECTURE FOR SURFACES WITH A $(\mathbb{Z}/2\mathbb{Z})^2$ -ACTION

The aim of this note is to prove Bloch’s conjecture for generalized Burniat type surfaces with  $p_g = q = 0$ . The proof uses the method of “enough automorphisms” introduced by Inose and Mizukami (cf. [IM79]) and refined by Barlow (cf. [Bar85]).

**Definition 1.1.** Let  $G$  be a finite group and  $H \leq G$  be a subgroup. Then we set

$$z(H) := \sum_{h \in H} h \in \mathbb{C}G$$

**Lemma 1.2** ([Bar85]). *Let  $S$  be a nonsingular surface and  $G$  a finite subgroup of  $\text{Aut}(S)$ . Let  $H, H_1, \dots, H_r$  be subgroups of  $G$ . We denote by  $\mathcal{I}$  the two-sided ideal of  $\mathbb{C}G$  generated by  $z(H_1), \dots, z(H_r)$ . Assume that*

$$i) \ z(H) \in \mathcal{I},$$

ii)  $T(S/H_i) = 0$ , for every  $i \in \{1, \dots, r\}$ .

Then  $T(S/H) = 0$ .

Using this result, the first author proved the following

**Proposition 1.3** ([Bau14, Proposition 1.3]). *Let  $S$  be a surface of general type with  $p_g(S) = 0$ . Assume that  $G = (\mathbb{Z}/2\mathbb{Z})^2 \triangleleft \text{Aut}(S)$ . Then  $S$  satisfies Bloch's conjecture if and only if for each  $\sigma \in G \setminus \{0\}$  the quotient  $S/\sigma$  satisfies Bloch's conjecture.*

*Remark 1.4.* Note that  $S/\sigma$  is a surface with at most nodes as singularities and denoting by  $X_\sigma \rightarrow S/\sigma$  the resolution of its singularities,  $X_\sigma$  is minimal and has  $p_g = 0$ . Moreover, since nodes are rational singularities,  $T(S/\sigma) = T(X_\sigma)$ .

Using the fact that by the result of Bloch, Kas and Liebermann (cf. [BKL76]) Bloch's conjecture is true for surfaces  $S$  with  $\text{kod}(S) \leq 1$ , we obtain the following:

**Corollary 1.5** ([Bau14, Corollary 1.5]). *Let  $S$  be a surface of general type with  $p_g(S) = 0$  and assume that  $G = (\mathbb{Z}/2\mathbb{Z})^2 \triangleleft \text{Aut}(S)$ . Assume that for each  $\sigma \in G \setminus \{0\}$  the quotient  $S/\sigma$  has  $\text{kod}(S/\sigma) \leq 1$ , then  $S$  satisfies Bloch's conjecture.*

## 2. INVOLUTIONS ON SURFACES OF GENERAL TYPE

In this section we collect some results regarding involutions on surfaces of general type, that we need in Section 4. We start fixing some notation.

Let  $S$  be a minimal regular surface of general type with an involution  $\sigma$ . Then  $\sigma$  is biregular and its fixed locus is the union of  $k$  isolated points  $P_1, \dots, P_k$  and a smooth (not necessarily connected) curve  $R$ . We denote by  $p: S \rightarrow \Sigma := S/\langle \sigma \rangle$  the projection onto the quotient, by  $B$  the image of  $R$  and by  $Q_j$  the image of  $P_j$ ,  $j = 1, \dots, k$ . The surface  $\Sigma$  is normal,  $Q_1, \dots, Q_k$  are nodes and they are the only singularities of  $\Sigma$ . Let  $h: V \rightarrow S$  be the blow-up of  $S$  at  $P_1, \dots, P_k$  and  $E_j$  be the exceptional curve over  $P_j$ ,  $j = 1, \dots, k$ .

The involution  $\sigma$  induces an involution  $\tilde{\sigma}$  on  $V$  whose fixed locus is the union of  $R' = h^{-1}R$  and of  $E_1, \dots, E_k$ . Let  $\pi: V \rightarrow W := V/\langle \tilde{\sigma} \rangle$  be the projection onto the quotient and set  $B' := \pi(R')$ ,  $A_j := \pi(E_j)$ ,  $j = 1, \dots, k$ . The surface  $W$  is smooth and the  $A_j$  are disjoint  $(-2)$ -curves. Let  $g$  be the morphism induced by  $h$ ,  $g$  is the minimal resolution of the singularities of  $\Sigma$  and we have the following commutative diagram:

$$(2.1) \quad \begin{array}{ccc} V & \xrightarrow{h} & S \\ \pi \downarrow & & \downarrow p \\ W & \xrightarrow{g} & \Sigma \end{array}$$

Let  $B'' := B' + \sum_{j=1}^k A_j$  be the branch divisor of  $\pi$ , one has  $\pi_*\mathcal{O}_V = \mathcal{O}_W \oplus \mathcal{O}_W(-\Delta')$  with  $B'' \equiv 2\Delta'$ .

Now let us assume that  $W$  is of general type. We denote by  $P$  the minimal model of  $W$ , by  $\rho: W \rightarrow P$  the corresponding projection and let  $\overline{B} := \rho_*(B'')$ . Then  $\overline{B}$  is an even divisor linearly equivalent to  $2\Delta$ , where  $\Delta := \rho_*(\Delta')$ .  $V$  is the canonical resolution of the double cover of  $P$  branched along  $\overline{B}$  and, by [Hor75, Lemma 6], one has:

$$(2.2) \quad K_S^2 - k = K_V^2 = 2(K_P + \Delta)^2 - 2 \sum_i (x_i - 1)^2,$$

$$(2.3) \quad \chi(\mathcal{O}_S) = \chi(\mathcal{O}_V) = 2\chi(\mathcal{O}_P) + \frac{1}{2}(K_P + \Delta) \cdot \Delta - \frac{1}{2} \sum_i x_i(x_i - 1),$$

where  $x_i := \lfloor \frac{m_i}{2} \rfloor$ , being  $m_i$  the multiplicity of the singular point  $b_i$  of  $\overline{B}$ . We recall the following:

**Proposition 2.1** ([Bom73, Proposition 1]). *Let  $S$  be a minimal surface of general type. Let  $C$  be an irreducible curve on  $S$ , then  $K_S \cdot C \geq 0$  and if  $K_S \cdot C = 0$ , then  $C^2 = -2$  and  $C$  is a rational non-singular curve.*

We can now prove:

**Proposition 2.2.** *Let  $S$  be a (minimal) surface of general type with  $K_S^2 = 6$ ,  $p_g(S) = 0$  and that contains no rational curves except at most a  $(-2)$ -curve  $L$ . Let  $\sigma$  be an involution on  $S$  such that one of the following holds:*

- (i) *either  $\text{Fix}(\sigma)$  contains more than 8 isolated points and a non-rational curve;*
- (ii) *or  $\text{Fix}(\sigma)$  is given by 6 isolated points and an elliptic curves  $C$  such that  $C \cap L = \emptyset$ ;*
- (iii) *or  $\text{Fix}(\sigma)$  is given by 4 isolated points, an elliptic curves  $C$  and the  $(-2)$ -curve  $L$ :  $C \cap L = \emptyset$ .*

*Then  $\Sigma := S/\langle \sigma \rangle$  is not of general type.*

*Proof.* We use the notation of above and aiming for a contradiction we assume  $W$  of general type.

Since  $S$  is a minimal surface of general type with  $K_S^2 = 6$  and  $p_g(S) = 0$ , then  $p_g(P) = q(P) = 0$  and from formulas (2.2), (2.3) we get

$$(2.4) \quad 5 - \frac{k}{2} = K_P^2 + K_P \cdot \Delta + \sum_i (x_i - 1).$$

We claim that  $K_P \cdot \overline{B} = 2K_P \cdot \Delta > 0$ , indeed:  $\overline{B} = \sum_{l=1}^r B_l + \sum_j \rho_* A_j$ , where, by assumption, each  $B_l$  is an irreducible curve,  $r > 0$  and  $B_t$  is non-rational for at least one  $t \in \{1, \dots, r\}$ . By Proposition 2.1,  $K_P \cdot B_l \geq 0$  for any  $l = 1, \dots, r$  and  $K_P \cdot B_t > 0$  and the claim follows.

*Case (i):*  $k \geq 8$  and by (2.4) we get:

$$1 \geq K_P^2 + K_P \cdot \Delta + \sum_i (x_i - 1),$$

but this is not possible because  $K_P^2 \geq 1$  and  $K_P \cdot \Delta > 0$ .

*Case (ii):*  $k = 6$  and by (2.4) we get:

$$2 = K_P^2 + K_P \cdot \Delta + \sum_i (x_i - 1),$$

therefore  $K_P^2 = K_P \cdot \Delta = 1$  and  $x_i = 1$  for any  $i$ .

Since  $e(V) = 12$ , it holds  $e(W) = 6 + \frac{1}{2}e(B'')$ ; by assumption  $B'' := \sum_{j=1}^6 A_j + \tilde{C}$  with  $\tilde{C} := \pi(h^{-1}(C))$  a smooth elliptic curve:  $e(B'') = 12$ .

By Noether's formula  $K_W^2 = 0$ , hence  $\rho$  is the blow-down of exactly one  $(-1)$ -curve  $E$ . In particular  $E$  does not intersect any  $(-2)$ -curves.

By assumption,  $S$  contains no rational curves except at most a  $(-2)$ -curve  $L$ , such that  $L \cap C = \emptyset$ . If  $S$  contains such a curve  $L$ , then  $\sigma$  maps  $L$  onto itself hence two  $\sigma$ -fixed points lie on it; an easy computation shows that  $\pi(\tilde{L})$  is a  $(-2)$ -curve on  $W$ , being  $\tilde{L} \subset V$  the strict transform of  $L$ . Therefore, the rational curve  $E$  must intersect  $\tilde{C}$  in at least 4 points (by Hurwitz' formula) and  $\overline{B}$  contains a singular point with  $m_i \geq 4$ , i.e.  $x_i \geq 2$ .

*Case (iii):*  $k = 4$  and by (2.4) we get:

$$3 = K_P^2 + K_P \cdot \Delta + \sum_i (x_i - 1),$$

therefore  $K_P^2 \leq 2$ .

Let  $\tilde{C} := \pi(h^{-1}(C))$  and  $\tilde{L} := \pi(h^{-1}(L))$ , then  $B'' = \sum_{j=1}^4 A_j + \tilde{C} + \tilde{L}$  is the disjoint union of five rational curves and an elliptic curve:  $e(B'') = 10$ . Since  $e(V) = 10$ ,  $e(W) = \frac{1}{2}e(V) + \frac{1}{2}e(B'') = 10$  and by Noether's formula  $2 = K_W^2 \leq K_P^2$ . We get that  $W = P$  is minimal,  $K_P \cdot \Delta = 1$  and  $x_i = 1$  for any  $i$ . By (2.3), it follows  $\Delta^2 = -3$  and so  $-12 = B''^2 = -12$ . It is direct to show that  $\tilde{L}$  is a  $(-4)$ -curve, hence  $\tilde{C}^2 - 12 = B''^2$ . We have an elliptic curve  $\tilde{C}$  with  $\tilde{C}^2 = 0$  on a minimal surface of general type, it contradicts Proposition 2.1.  $\square$

### 3. GENERALIZED BURNIAT TYPE SURFACES

In this section we recall the construction of generalized Burniat type surfaces. For further details we refer to [BCF].

For  $j = 1, 2, 3$ , let  $E_j = \mathbb{C}/\langle 1, \tau_j \rangle$  be an elliptic curve and denote by  $z_j$  a uniformizing parameter on  $E_j$ . Let  $\mathcal{L}_j$  be the Legendre  $\mathcal{L}$ -function for  $E_j$ :  $\mathcal{L}_j$  is a meromorphic function on  $E_j$  and  $\mathcal{L}_j: E_j \rightarrow \mathbb{P}^1$  is a double cover branched over four distinct points:  $\pm 1, \pm a_j \in \mathbb{P}^1 \setminus \{0, \infty\}$ . It is well known that the following statements hold (see [Ino94, Lemma 3-2] and [BC11, Section 1]):

- $\mathcal{L}_j(0) = 1$ ,  $\mathcal{L}_j(\frac{1}{2}) = -1$ ,  $\mathcal{L}_j(\frac{\tau_j}{2}) = a_j$ ,  $\mathcal{L}_j(\frac{\tau_j+1}{2}) = -a_j$ ;

- let  $b_j := \mathcal{L}_j(\frac{\tau_j}{4})$ , then  $b_j^2 = a_j$ ;
- $\frac{d\mathcal{L}_j}{dz_j}(z_j) = 0$  if and only if  $z_j \in \left\{0, \frac{1}{2}, \frac{\tau_j}{2}, \frac{\tau_j+1}{2}\right\}$  (since these are the ramification points of  $\mathcal{L}_j$ );
- $\mathcal{L}_j(z_j) = \mathcal{L}_j(z_j + 1) = \mathcal{L}_j(z_j + \tau_j) = \mathcal{L}_j(-z_j) = -\mathcal{L}_j(z_j + \frac{1}{2})$ ;
- $\mathcal{L}_j\left(z_j + \frac{\tau_j}{2}\right) = \frac{a_j}{\mathcal{L}_j(z_j)}$ .

For  $j \in \{1, 2, 3\}$ , we define an action of  $\{(\zeta_j, \eta_j, \epsilon_j)\} \cong (\mathbb{Z}/2\mathbb{Z})^3$  on  $E_j$ , as follows:

$$(3.1) \quad \begin{aligned} (z_j \mapsto -z_j) &\hat{=} (1, 0, 0) \\ (z_j \mapsto -z_j + \frac{\tau_j}{2}) &\hat{=} (0, 1, 0) \\ (z_j \mapsto -z_j + \frac{1}{2}) &\hat{=} (0, 0, 1), \end{aligned}$$

and we consider the induced action of  $\mathcal{G} = \{(\zeta_1, \eta_1, \epsilon_1, \zeta_2, \eta_2, \epsilon_2, \zeta_3, \eta_3, \epsilon_3)\}$  on  $T := E_1 \times E_2 \times E_3$ . We define also the following map:

$$(3.2) \quad \begin{aligned} \pi' : E_1 \times E_2 \times E_3 &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ (z_1, z_2, z_3) &\longmapsto \left( \frac{\mathcal{L}_1(z_1)}{b_1}, \frac{\mathcal{L}_2(z_2)}{b_2}, \frac{\mathcal{L}_3(z_3)}{b_3} \right). \end{aligned}$$

The  $\mathcal{G}$ -action on  $T$  induces, via  $\pi'$ , an action of  $\mathcal{H} := (\mathcal{H}_1)^3 \cong (\mathbb{Z}/2\mathbb{Z})^6$  on  $P_1 := (\mathbb{P}^1)^3$ , where  $\mathcal{H}_1 \cong (\mathbb{Z}/2\mathbb{Z})^2$  acts on  $\mathbb{P}^1$  in this way:

$$(3.3) \quad \begin{aligned} (1, 0) &\hat{=} ((s : t) \mapsto (t : s)), \\ (0, 1) &\hat{=} ((s : t) \mapsto (s : -t)), \end{aligned}$$

being  $(s : t)$  homogeneous coordinates of  $\mathbb{P}^1$ .

Let  $Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  be an irreducible Del Pezzo surface of degree 6 invariant under a subgroup  $H \cong (\mathbb{Z}/2\mathbb{Z})^2 \triangleleft \mathcal{H}$ .

The inverse image  $\hat{X} := \pi'^{-1}(Y)$  of  $Y$  under  $\pi'$  is an irreducible hypersurface in the product of three smooth elliptic curves  $T := E_1 \times E_2 \times E_3$ , which is of multi degree  $(2, 2, 2)$ .

**Definition 3.1.**  $\hat{X}$  is called a *Burniat hypersurface* in  $T$ .

*Remark 3.2.* According to [BCF], every Burniat hypersurface is given by one of the following equations:

$$(3.4) \quad \begin{aligned} \hat{X}_\nu = \{(z_1, z_2, z_3) \in T \mid \nu_1(\mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) + b_1b_2b_3) + \\ \nu_2(\mathcal{L}_1(z_1)b_2b_3 + b_1\mathcal{L}_2(z_2)\mathcal{L}_3(z_3)) = 0\}, \end{aligned}$$

$$(3.5) \quad \hat{X}_\mu = \{(z_1, z_2, z_3) \in T \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = \mu\},$$

$$(3.6) \quad \hat{X}_b = \{(z_1, z_2, z_3) \in T \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = b_1b_2b_3\},$$

where  $\nu := (\nu_1 : \nu_2) \in \mathbb{P}^1$ ,  $\mu \in \mathbb{C}$  and  $b := b_1b_2b_3$ .

Recall that we are considering only values of  $\nu$  (resp.  $\mu$ ) such that  $\hat{X}_\nu$  (resp.  $\hat{X}_\mu$ ) is irreducible, i.e.  $(\nu_1/\nu_2) \neq \pm 1$  and  $\mu \neq 0$ .

*Remark 3.3.* By construction, a Burniat hypersurface  $\hat{X}$  has at most finitely many nodes as singularities. Therefore, denoting by  $\epsilon: X' \rightarrow \hat{X}$  the minimal resolution of its singularities, we have that  $K_{X'} = \epsilon^* K_{\hat{X}}$  and  $X'$  is a minimal surface of general type with  $K_{X'}^2 = 48$  and  $\chi(X') = 8$ .

Let  $\mathcal{G}_0 \cong (\mathbb{Z}/2\mathbb{Z})^6 \triangleleft \mathcal{G} \cong (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3$  be the group:

$$\mathcal{G}_0 := \{(\zeta_1, \eta_1, \epsilon_1, \zeta_2, \eta_2, \epsilon_2, \zeta_3, \eta_3, \epsilon_3) \mid \eta_1 = \eta_2 = \eta_3, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0\}.$$

Then we have the following:

**Lemma 3.4** ([BCF]). (1)  $\hat{X}_\nu$  is invariant under the group

$$\mathcal{G}'_1 := \{(\zeta_1, \eta_1, 0, \zeta_2, \eta_1, \epsilon_2, \zeta_3, \eta_1, \epsilon_3) \mid \epsilon_2 + \epsilon_3 = 0\} \cong (\mathbb{Z}/2\mathbb{Z})^5 \triangleleft \mathcal{G}_0.$$

(2)  $\hat{X}_\mu$  is invariant under the group

$$\mathcal{G}_1 := \{(\zeta_1, 0, \epsilon_1, \zeta_2, 0, \epsilon_2, \zeta_3, 0, \epsilon_3) \mid \epsilon_1 + \epsilon_2 + \epsilon_3 = 0\} \cong (\mathbb{Z}/2\mathbb{Z})^5 \triangleleft \mathcal{G}_0.$$

(3)  $\hat{X}_b$  is invariant under  $\mathcal{G}_0$ .

*Remark 3.5.* Let  $g \in \mathcal{G}_0 \setminus \{0\}$  be an element fixing points on  $T$ . By [BC13, Proposition 4.3],  $g$  is then an element in Table 1.

TABLE 1. The element of  $\mathcal{G}_0$  having fixed points on  $T$

	$g_1$	$g_2$	$g_3$	$g_4$	$g_5$	$g_6$	$g_7$	$g_8$	$g_9$	$g_{10}$	$g_{11}$	$g_{12}$	$g_{13}$	$g_{14}$	$g_{15}$	$g_{16}$	$g_{17}$
$\zeta_1$	0	0	1	0	1	1	0	0	0	1	1	0	0	0	0	1	1
$\eta_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
$\epsilon_1$	0	0	0	0	0	0	0	1	1	0	0	1	1	0	0	1	1
$\zeta_2$	0	1	0	1	0	1	0	0	0	1	0	1	0	0	1	0	1
$\eta_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
$\epsilon_2$	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1
$\zeta_3$	1	0	0	1	1	0	0	0	0	1	0	0	1	0	1	1	0
$\eta_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
$\epsilon_3$	0	0	0	0	0	0	1	1	0	0	1	1	0	0	1	1	0

1) Let  $\hat{X} := \hat{X}_b$ . In Table 1, the elements  $g_1$ - $g_3$  fix pointwise a surface  $S \subset T$ . Each element  $g_4$ - $g_9$  fixes pointwise a curve  $C \subset T$  and its fixed locus has non trivial intersection with  $\hat{X}$  since  $\hat{X} \subset T$  is an ample divisor. Finally, the elements  $g_{10}$ - $g_{17}$  have isolated fixed points on  $T$ ; in particular, the elements  $g_{11}$ - $g_{17}$  have fixed points on  $\hat{X}$ , while the fixed locus of element  $g_{10}$  intersects  $\hat{X}$  only for special choices of the three elliptic curves.

2) The same holds for  $\hat{X}_\nu := \pi'^{-1}(Y_\nu)$  (resp.  $\pi'^{-1}(Y_\mu)$ ), considering only the elements  $g_1$ - $g_7, g_{10}, g_{11}, g_{14}, g_{15}$  (resp.  $g_1$ - $g_{13}$ ), i.e. the ones belonging to  $\mathcal{G}'_1$  (resp.  $\mathcal{G}_1$ ). In particular, the fixed locus of element  $g_{10}$  intersects  $\hat{X}$  only for special choices of the three elliptic curves and the parameter  $\nu$  (resp.  $\mu$ ).

**Definition 3.6.** Let  $\hat{X}$  be a Burniat hypersurface in  $E_1 \times E_2 \times E_3$  and let  $G \cong (\mathbb{Z}/2\mathbb{Z})^3$  be a subgroup of  $\mathcal{G}_0$  acting freely on  $\hat{X}$ . The minimal resolution  $S$  of the quotient surface  $\hat{S} := \hat{X}/G$  is called a *generalized Burniat type (GBT) surface*. We call  $\hat{S}$  the *quotient model of  $S$* .

*Remark 3.7.* 1) Since  $G$  acts freely and  $\hat{X}$  has at most nodes as singularities,  $\hat{S}$  is singular if and only if  $\hat{X}$  is singular and  $\hat{S}$  has at most nodes as singularities.

2) A generalized Burniat type surface  $S$  is a smooth minimal surface of general type with  $K_S^2 = 6$  and  $\chi(S) = 1$ .

In [BCF], GBT surfaces have been completely classified. In particular, it has been shown there are exactly four families of GBT surfaces with  $p_g = q = 0$ :

**Theorem 3.8.** *Let  $S \rightarrow \hat{S} = \hat{X}/G$  be a regular generalized Burniat type surface  $S$  then  $(\hat{X}, G) \in \{(\hat{X}_\nu, G_1), (\hat{X}_\mu, G_2), (\hat{X}_b, G_j), j = 3, 4\}$ , where the groups  $G_1, G_2, G_3, G_4$  are in Table 2.*

	$\zeta_1$	$\eta_1$	$\epsilon_1$	$\zeta_2$	$\eta_2$	$\epsilon_2$	$\zeta_3$	$\eta_3$	$\epsilon_3$
$G_1$	1	0	0	1	0	0	1	0	0
	0	1	0	1	1	0	1	1	0
	0	0	0	0	0	1	1	0	1
$G_2$	1	0	0	0	0	1	1	0	1
	0	0	1	0	0	0	1	0	1
	0	0	0	1	0	1	0	0	1
$G_3$	1	0	0	0	0	1	1	0	1
	0	1	0	0	1	0	1	1	0
	0	0	1	1	0	1	1	0	0
$G_4$	1	0	1	0	0	1	1	0	0
	0	1	0	0	1	0	1	1	0
	0	0	0	1	0	1	1	0	1

TABLE 2. The groups  $G_j$

To fix the notation, let us call a surface  $S$  a *generalized Burniat type (GBT) surface of type  $j$*  if  $S$  belongs to the (uniquely determined) family number  $j$  in Tables 2.

*Remark 3.9.* As shown in [BCF] GBT surfaces of type  $j$  ( $1 \leq j \leq 4$ ) have pairwise non isomorphic fundamental groups. In particular, they belong to different connected components of the moduli space of surfaces of general type.

*Remark 3.10.* Let  $\hat{S} := \hat{X}/G$  be the quotient model of a regular GBT surface  $S$ . According to Theorem 3.8, there are the following possibilities:



- a)  $\hat{X} = \hat{X}_\nu := \{(z_1, z_2, z_3) \in T \mid \nu_1 L_1(z_1, z_2, z_3) + \nu_2 L_2(z_1, z_2, z_3) = 0\}$   
and  $G = G_1$ , where  $\nu := (\nu_1 : \nu_2) \in \mathbb{P}^1$ ,  $(\nu_1/\nu_2) \neq \pm 1$ ,

$$L_1(z_1, z_2, z_3) := \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) + b_1 b_2 b_3 \quad \text{and}$$

$$L_2(z_1, z_2, z_3) := \mathcal{L}_1(z_1)b_2 b_3 + b_1 \mathcal{L}_2(z_2)\mathcal{L}_3(z_3).$$

Note that  $g_0 := (1, 0, 0, 1, 0, 0, 1, 0, 0) \in G_1$ , and its fixed locus  $F_0 = \{(z_1, z_2, z_3) \in T \mid 2z_1 = 2z_2 = 2z_3 = 0\}$  intersects  $\hat{X}_\nu$  if

$$\nu \in B_0 := \{(L_2(z_1, z_2, z_3) : -L_1(z_1, z_2, z_3)) \mid (z_1, z_2, z_3) \in F_0\}.$$

In other words, if  $\nu \in B_0$  then  $G_1$  does not act freely on  $\hat{X}_\nu$  and therefore does not give rise to a GBT surface.

- b)  $\hat{X} = \hat{X}_\mu := \{(z_1, z_2, z_3) \in T \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = \mu\}$ , with  $\mu \in \mathbb{C}$ ,  $\mu \neq 0$  and  $G = G_2$ . Since  $g_0 \in G_2$ , if  $\mu \in B' := \{\mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) \mid (z_1, z_2, z_3) \in F_0\}$ , then  $G_2$  does not act freely on  $\hat{X}_\mu$ ; moreover (see e.g. [Ino94]):

$$B' = \{\pm 1, \pm a_i, \pm a_i a_j, \pm a_1 a_2 a_3\} \quad \text{with } i \neq j \in \{1, 2, 3\}.$$

We remark that this case gives rise to the family of *primary Burniat surfaces* (see [BC11, BC13]).

- c)  $\hat{X} = \hat{X}_b := \{(z_1, z_2, z_3) \in T \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = b\}$ , with  $b := b_1 b_2 b_3$  and  $G = G_j$ ,  $j = 3, 4$ .

We already remarked that  $\hat{X}$  has at most finitely many nodes as singularities. The next statement shows that either  $\hat{X}$  is smooth or has exactly eight nodes.

**Proposition 3.11.** *Let  $S \rightarrow \hat{S} = \hat{X}/G$  be a regular generalized Burniat type surface, i.e.,  $(\hat{X}, G) \in \{(\hat{X}_\nu, G_1), (\hat{X}_\mu, G_2), (\hat{X}_b, G_j), j = 3, 4\}$ . Then:*

- 1)  $\hat{X}_\nu$  is singular if and only if  $\nu \in B := \{(\pm b_1 : 1), (1 : \pm b_1)\}$  and  $\text{Sing}(\hat{X}_\nu) = \{(z_1, \pm \frac{1}{4}, \pm \frac{1}{4}) \mid 2z_1 = 0, \nu b_1 + \mathcal{L}_1(z_1) = 0\} \cup \{(z_1, \frac{\tau_2}{2} \pm \frac{1}{4}, \frac{\tau_3}{2} \pm \frac{1}{4}) \mid 2z_1 = 0, b_1 + \nu \mathcal{L}_1(z_1) = 0\}$ ;
- 2)  $\hat{X}_\mu$  is smooth;
- 3-4)  $\hat{X}_b$  is singular if and only if  $b := b_1 b_2 b_3 \in B'$  and  $\text{Sing}(\hat{X}_b) = \{(z_1, z_2, z_3) \in \hat{X}_b \mid 2z_1 = 2z_2 = 2z_3 = 0\}$ .

In particular, either  $\hat{X}$  is smooth or its singular locus consists of exactly 8 nodes.

*Proof.* We start with cases 2) and 3-4).

Let  $f(z_1, z_2, z_3) := \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3)$ . It is easy to see that

$$\text{Sing}(\hat{X}_\mu) = \{z \in \hat{X}_\mu \mid \nabla f(z) = 0\} = \{(z_1, z_2, z_3) \in \hat{X}_\mu \mid 2z_1 = 2z_2 = 2z_3 = 0\}$$

since  $\frac{df}{dz_i} = \frac{d\mathcal{L}_i}{dz_i} \mathcal{L}_{i+1} \mathcal{L}_{i+2}$  (the indices  $i \in \{1, 2, 3\}$  have to be considered mod 3). We observe that  $\text{Sing}(\hat{X}_\mu) = \hat{X}_\mu \cap F_0$ , being  $F_0$  the fixed locus of  $g_0 := (1, 0, 0, 1, 0, 0, 1, 0, 0)$ .

Since  $g_0 \in G_2$ , we get that the surfaces of case 2) are smooth.

In case 3-4),  $\mu := b = b_1 b_2 b_3$  and  $\hat{X}_b$  is singular if and only if  $b \in B' = \{f(z) \mid z \in F_0\}$ ; we have to determine the number of nodes. We note that if  $(z_1, z_2, z_3) \in F_0$  then  $f(z_1, z_2, z_3) = f(z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3) = f(z_1, z_2 + \frac{1}{2}, z_3 + \frac{1}{2}) = f(z_1 + \frac{1}{2}, z_2, z_3 + \frac{1}{2})$ , and all these points are in  $F_0$ . Since  $a_j = b_j^2 \neq \pm 1$  and  $\hat{X}_b$  is invariant under  $(*, 1, *, *, 1, *, *, 1, *)$ , it is easy to show that  $b = \pm 1$  iff  $b = \pm a_1 a_2 a_3$  and that  $b = \pm a_i$  iff  $b = \pm a_{i+1} a_{i+2}$  (indices have to be considered mod 3); in particular,  $|B'| = 8$ . Since  $|F_0| = 64$ , it follows that for any choice of  $b \in B'$  exactly 8 points of  $F_0$  belong to  $\hat{X}_b$ , i.e.  $\text{Sing}(\hat{X}_b)$  consist of exactly 8 nodes.

In case 1), let  $f_\nu(z_1, z_2, z_3) := \nu_1 L_1(z_1, z_2, z_3) + \nu_2 L_2(z_1, z_2, z_3)$ . We note that if  $\nu_1 = 0$  or  $\nu_2 = 0$ , arguing as above we get that  $\hat{X}_\nu$  is smooth ( $g_0 \in G_1$ ), so we may assume  $\nu_2 = 1$  and  $\nu := \nu_1 \in \mathbb{C} \setminus \{0\}$ . We recall that  $\mathcal{L}_i$  has poles in  $z_i = \frac{\tau_i}{2} \pm \frac{1}{4}$  and zeroes in  $z_i = \pm \frac{1}{4}$ . We start considering charts such that  $z_i \neq \frac{\tau_i}{2} \pm \frac{1}{4}$  for  $i = 1, 2, 3$ . It is easy to see that

$$(3.7) \quad \nabla f_\nu = 0 \iff \begin{cases} \mathcal{L}'_1(\nu \mathcal{L}_2 \mathcal{L}_3 + b_2 b_3) = 0 \\ \mathcal{L}'_2 \mathcal{L}_3(\nu \mathcal{L}_1 + b_1) = 0 \\ \mathcal{L}_2 \mathcal{L}'_3(\nu \mathcal{L}_1 + b_1) = 0 \end{cases} \quad \text{with} \quad \mathcal{L}'_i = \frac{d\mathcal{L}_i}{dz_i}.$$

If  $\nu \mathcal{L}_1 + b_1 = 0$  for a point in  $\hat{X}$ , then

$$f_\nu = \mathcal{L}_2 \mathcal{L}_3(\nu \mathcal{L}_1 + b_1) + b_2 b_3(\nu b_1 + \mathcal{L}_1) = \nu b_1 + \mathcal{L}_1 = 0$$

hence  $\nu = \pm 1$ , i.e.  $\hat{X}$  is not irreducible, a contradiction (see Remark 3.2); analogously we can assume  $\nu \mathcal{L}_2 \mathcal{L}_3 + b_2 b_3 \neq 0$ .

Since  $\mathcal{L}_i$  and  $\mathcal{L}'_i$  have no common zeroes,  $\mathcal{L}'_2 = 0$  if and only if  $\mathcal{L}'_3 = 0$ ; in this case the solutions of  $\nabla f_\nu = 0$  are points in  $F_0$ , the fixed locus of  $g_0 \in G_1$ . Therefore  $(z_1, z_2, z_3) \in \text{Sing}(\hat{X})$  if and only if it satisfies the following equations:  $\mathcal{L}_2(z_2) = \mathcal{L}_3(z_3) = 0$ ,  $\mathcal{L}'_1(z_1) = 0$  and  $f_\nu = \nu b_1 + \mathcal{L}_1(z_1) = 0$ . It is immediate to see that the last two equations have common solutions if and only if  $\nu \in B = \{\pm b_1, \pm b_1^{-1}\}$ ; if  $\nu \in B$ , we find 4 nodes:  $z_1 = \mathcal{L}_1^{-1}(-\nu b_1)$ ,  $z_2 \in \{\pm \frac{1}{4}\}$  and  $z_3 \in \{\pm \frac{1}{4}\}$ .

We now consider charts such that  $z_2 \neq \pm \frac{1}{4}$  and  $z_3 \neq \pm \frac{1}{4}$  then the affine equation  $f_\nu = 0$  can be written as follows:

$$f_\nu = \overline{\mathcal{L}}_2 \overline{\mathcal{L}}_3 b_2 b_3(\nu b_1 + \mathcal{L}_1) + (\nu \mathcal{L}_1 + b_1)$$

being  $\overline{\mathcal{L}}_i := \mathcal{L}_i^{-1}$ ,  $i = 2, 3$ . Arguing as above, one gets that  $(z_1, z_2, z_3) \in \text{Sing}(\hat{X})$  if and only if it satisfies the following equations:  $\overline{\mathcal{L}}_2(z_2) = \overline{\mathcal{L}}_3(z_3) = 0$ ,  $\mathcal{L}'_1(z_1) = 0$  and  $f_\nu = \nu \mathcal{L}_1(z_1) + b_1 = 0$ . The last two equations have common solutions if and only if  $\nu \in B = \{\pm b_1, \pm b_1^{-1}\}$ ; if  $\nu \in B$ , we find other 4 nodes, namely:  $z_1 = \mathcal{L}_1^{-1}(-\frac{b_1}{\nu})$ ,  $z_2 \in \{\frac{\tau_2}{2} \pm \frac{1}{4}\}$  and  $z_3 \in \{\frac{\tau_3}{2} \pm \frac{1}{4}\}$ .

Considering the other charts, one finds either no singular points, or four of the eight nodes we found.  $\square$

**Corollary 3.12.** *Let  $S \rightarrow \hat{S} := \hat{X}/G$  be a generalized Burniat surface. Then either  $\hat{S}$  is smooth or its singular locus is given by exactly one node.*

*Proof.* We simply note that  $G \cong (\mathbb{Z}/2\mathbb{Z})^3$  acts freely on  $\hat{X}$ ; in particular it acts transitively on the set of nodes.  $\square$

#### 4. THE MAIN RESULT

In this section we give a prove (using Corollary 1.5 and Proposition 2.2) that Bloch's conjecture holds for regular generalized Burniat type surfaces.

*Remark 4.1.* Let  $S \rightarrow \hat{S} := \hat{X}/G$  be a GBT and let  $\gamma: \hat{X} \rightarrow \hat{S} := \hat{X}/G$  be the projection onto the quotient. Let  $\sigma$  be an involution on  $\hat{X}$ , it defines an involution  $\bar{\sigma}$  on  $\hat{S}$ :  $\bar{\sigma}(\gamma(x)) := \gamma(\sigma(x))$  and

$$\text{Fix}(\bar{\sigma}) = \bigcup_{g \in G} \gamma(\text{Fix}_{\hat{X}}(\sigma g)),$$

being  $\text{Fix}_{\hat{X}}(\sigma) := \text{Fix}(\sigma) \cap \hat{X}$ . Moreover,  $\bar{\sigma}$  lifts to an involution  $\sigma' := \epsilon^{-1} \circ \bar{\sigma} \circ \epsilon$  on  $S$ .

Generalized Burniat type surfaces are constructed considering  $G \cong (\mathbb{Z}/2\mathbb{Z})^3 \triangleleft \mathcal{G}_0$  acting freely on a Burniat hypersurface  $\hat{X} \subset T$ , hence it is natural to consider involutions in  $\mathcal{G}_0 \setminus G$ . We start determining the fixed locus of elements in  $\mathcal{G}_0$ .

**Lemma 4.2.** *Let  $S \rightarrow \hat{S} = \hat{X}/G$  be a regular generalized Burniat type surface with  $(\hat{X}, G) = (\hat{X}_\nu, G_1)$ . Let  $g \in \mathcal{G}'_1$  be an element fixing point on  $X_\nu$ , then its fixed locus  $\text{Fix}(g)$  on  $\hat{X}_\nu$  is as in Table 3.*

	$\text{Fix}(g_i), X_\nu \text{ smooth}$	$\text{Fix}(g_i), X_\nu \text{ singular}$
$g_1 := (0, 0, 0, 0, 0, 0, 1, 0, 0)$	4 genus 5 curves	4 genus 5 curves
$g_2 := (0, 0, 0, 1, 0, 0, 0, 0, 0)$		$\Gamma$
$g_3 := (1, 0, 0, 0, 0, 0, 0, 0, 0)$		
$g_4 := (0, 0, 0, 1, 0, 0, 1, 0, 0)$	32 pt	32 pt
$g_5 := (1, 0, 0, 0, 0, 0, 1, 0, 0)$		
$g_6 := (1, 0, 0, 1, 0, 0, 0, 0, 0)$		
$g_7 := (0, 0, 0, 0, 0, 1, 0, 0, 1)$	16 pt, 8 ell. curves	8 nodes, 8 ell. curves
$g_{11} := (1, 0, 0, 0, 0, 1, 0, 0, 1)$	32 pt	32 pt, 8 nodes
$g_{14} := (0, 1, 0, 0, 1, 0, 0, 1, 0)$		32 pt
$g_{15} := (0, 1, 0, 1, 1, 1, 1, 1, 1)$		

TABLE 3.

In Table 3,  $\Gamma$  denotes the disjoint union  $\Gamma := C_1 \sqcup C_2 \sqcup D$ , being  $C_i$  ( $i = 1, 2$ ) a genus 5 curve and  $D$  the union of 8 elliptic curves each one passing through exactly two nodes and such that a point belongs to two of them if and only if it is a node.

*Proof.* By Proposition 3.11,  $\hat{X}_\nu$  is singular if and only if  $\nu \in B := \{(\pm b_1 : 1), (1 : \pm b_1)\}$ ; in this case the eight nodes on  $\hat{X}_\nu$  are fixed by  $g_{11} := (1, 0, 0, 0, 0, 1, 0, 0, 1)$ .

$g_1$ : since  $\text{Fix}(g_1) \cap \text{Fix}(g_{11}) = \emptyset$ , the fixed locus of  $g_1$  is independent from  $\nu$ . It fixes the points  $(z_1, z_2, \bar{z}_3)$  with  $2\bar{z}_3 = 0$ :  $\bar{z}_3 \in \{0, \frac{1}{2}, \frac{\tau_3}{2}, \frac{\tau_3+1}{2}\}$ , hence it cuts on  $\hat{X}_\nu$  four disjoint curves of genus 5: each one is given by an equation of multidegree (2,2) in  $E_1 \times E_2$ .

$g_2$ : this case is analogous to the previous one.

$g_3$ : it fixes the points with  $2z_1 = 0$  that is  $z_1 \in V := \{0, \frac{1}{2}, \frac{\tau_1}{2}, \frac{\tau_1+1}{2}\}$ .

If  $\hat{X}_\nu$  is smooth, this case is analogous to  $g_1$ : we get four disjoint smooth curves of genus 5 on  $\hat{X}_\nu$ .

If  $\hat{X}_\nu$  is singular ( $\nu \in B$ ), we rewrite the equation of  $\hat{X}_\nu$  as follows:

$$\mathcal{L}_2(z_2)\mathcal{L}_3(z_3)(\nu\mathcal{L}_1(z_1) + b_1) + b_2b_3(\nu b_1 + \mathcal{L}_1(z_1)) = 0.$$

For a fixed value  $\nu \in B$ , there exists a unique  $\bar{z}_1 \in V$  such that  $\nu b_1 + \mathcal{L}_1(\bar{z}_1) = 0$ , and the equation of  $\hat{X}_\nu$  is satisfied if and only if  $\mathcal{L}_2(z_2) = 0$  or  $\mathcal{L}_3(z_3) = 0$ . We get four elliptic curves on  $\hat{X}_\nu$  fixed by  $g_3$ :  $(\bar{z}_1, \pm\frac{1}{4}, z_3)$ ,  $(\bar{z}_1, z_2, \pm\frac{1}{4})$ . Analogously, considering the element  $z'_1 := \bar{z}_1 + \frac{\tau_1}{2} \in V$  we get other four elliptic curves  $(z'_1, \frac{\tau_1}{2} \pm \frac{1}{4}, z_3)$  and  $(z'_1, z_2, \frac{\tau_1}{2} \pm \frac{1}{4})$  on  $\hat{X}_\nu$  fixed by  $g_3$ . We observe that that a point belongs to two of these eight curves if and only if it is a node.

Considering  $z_1 \in V \setminus \{\bar{z}_1, z'_1\}$ , we get two disjoint curves of genus 5: both given by an equation of multidegree (2,2) in  $E_2 \times E_3$ .

$g_4$ : since  $\text{Fix}(g_4) \cap \text{Fix}(g_{11}) = \emptyset$ , the fixed locus of  $g_4$  is independent from  $\nu$ . It fixes the points  $(z_1, \bar{z}_2, \bar{z}_3)$  with  $2\bar{z}_2 = 2\bar{z}_3 = 0$  and  $2z_1 \neq 0$ , since  $g_0 := (1, 0, 0, 1, 0, 0, 1, 0, 0) \in G_1$  has no fixed points: for any pair  $(\bar{z}_2, \bar{z}_3)$ , the equation defining  $\hat{X}_\nu$  has two distinct solutions, whence  $g_4$  fixes 32 points on  $\hat{X}_\nu$ .

$g_5$ - $g_6$ : these cases are analogous to  $g_4$ .

$g_7$ : The involution  $g_7$  fixes the points  $(z_1, \bar{z}_2, \bar{z}_3)$  with  $2\bar{z}_2 = \frac{1}{2}$ ,  $2\bar{z}_3 = \frac{1}{2}$ . Let  $(\mathcal{L}_i(z_i)_0 : \mathcal{L}_i(z_i)_1)$  be the homogeneous coordinates of the point  $\mathcal{L}_i(z_i)$ . The equation of  $\hat{X}_\nu$  is then

$$\begin{aligned} & \nu[\mathcal{L}_1(z_1)_0\mathcal{L}_2(z_2)_0\mathcal{L}_3(z_3)_0 + b_1b_2b_3\mathcal{L}_1(z_1)_1\mathcal{L}_2(z_2)_1\mathcal{L}_3(z_3)_1] + \\ & [b_2b_3\mathcal{L}_1(z_1)_0\mathcal{L}_2(z_2)_1\mathcal{L}_3(z_3)_1 + b_1\mathcal{L}_2(z_2)_0\mathcal{L}_3(z_3)_0\mathcal{L}_1(z_1)_1] = 0. \end{aligned}$$

It follows easily from the properties of the Legendre  $\mathcal{L}$ -function that  $(\mathcal{L}_i(\frac{1}{4})_0 : \mathcal{L}_i(\frac{1}{4})_1) = (0 : 1)$ ,  $(\mathcal{L}_i(\frac{1}{4} + \frac{\tau_i}{2})_0 : \mathcal{L}_i(\frac{1}{4} + \frac{\tau_i}{2})_1) = (1 : 0)$ . If  $z_2 = \pm\frac{1}{4}$  and  $z_3 = \pm\frac{1}{4} + \frac{\tau_3}{2}$  or  $z_2 = \pm\frac{1}{4} + \frac{\tau_2}{2}$  and  $z_3 = \pm\frac{1}{4}$ , then the

equation is satisfied for any  $z_1 \in E_1$ , i.e.  $g_7$  fixes 8 disjoint elliptic curves contained in the smooth locus of  $\hat{X}_\nu$ .

If  $z_2 = \pm\frac{1}{4}$  and  $z_3 = \pm\frac{1}{4}$  then the equation becomes  $\nu b_1 \mathcal{L}_1(z_1)_1 + \mathcal{L}_1(z_1)_0 = 0$  that has two solutions if  $\nu \notin B$  (i.e.  $\hat{X}_\nu$  is smooth) and one solution if  $\nu \in B$ ; in other words, if  $\hat{X}_\nu$  is smooth  $g_4$  fixes 8 isolated points, else  $g_7$  fixes 4 nodes. Analogously, if  $z_2 = \pm\frac{1}{4} + \frac{\tau_2}{2}$  and  $z_3 = \pm\frac{1}{4} + \frac{\tau_3}{2}$  and if  $\hat{X}_\nu$  is smooth  $g_7$  fixes other 8 isolated points, else it fixes the other 4 nodes.

$g_{11}$ : we observe that  $\text{Fix}(g_{11}) = \text{Fix}(g_7) \cap \{2z_1 = 0\}$  and arguing as above we distinguish three cases: if  $z_2 = \pm\frac{1}{4}$  and  $z_3 = \pm\frac{1}{4} + \frac{\tau_3}{2}$  or  $z_2 = \pm\frac{1}{4} + \frac{\tau_2}{2}$  and  $z_3 = \pm\frac{1}{4}$ , then the equation of  $\hat{X}_\nu$  is satisfied for any  $z_1 \in E_1$ , but  $2z_1 = 0$  hence  $g_{11}$  fixes 32 smooth points on  $\hat{X}_\nu$ .

If  $z_2 = \pm\frac{1}{4}$  and  $z_3 = \pm\frac{1}{4}$  then the equation of  $\hat{X}_\nu$  is  $\nu b_1 \mathcal{L}_1(z_1)_1 + \mathcal{L}_1(z_1)_0 = 0$ , since  $2z_1 = 0$  we get no solution if  $\hat{X}_\nu$  is smooth ( $\nu \notin B$ ) and one solution (a node) if  $\hat{X}_\nu$  is singular ( $\nu \in B$ ).

An analogous argument holds if  $z_2 = \pm\frac{1}{4} + \frac{\tau_2}{2}$  and  $z_3 = \pm\frac{1}{4} + \frac{\tau_3}{2}$ .

Therefore, if  $\hat{X}_\nu$  is smooth  $g_{11}$  fixes 32 isolated points, else  $g_{11}$  fixes 32 smooth isolated points and 8 nodes.

$g_{14}$ : since  $\text{Fix}(g_{14}) \cap \text{Fix}(g_{11}) = \emptyset$ , the fixed locus of  $g_{14}$  is independent from  $\nu$ . It fixes 64 points on  $E_1 \times E_2 \times E_3$ , namely

$$z \in \left\{ \frac{1}{4} \begin{pmatrix} \pm\tau_1 \\ \pm\tau_2 \\ \pm\tau_3 \end{pmatrix} + \frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^3 \right\}.$$

Observe that  $\mathcal{L}_k(\pm\frac{\tau_k}{4}) = b_k$  and  $\mathcal{L}_k(\pm\frac{\tau_k}{4} + \frac{1}{2}) = -b_k$ . It is a straightforward computation to show that exactly 32 of them lie on  $\hat{X}_\nu$ .

$g_{15}$ : since  $\text{Fix}(g_{15}) \cap \text{Fix}(g_{11}) = \emptyset$ , the fixed locus of  $g_{15}$  is independent from  $\nu$ . It fixes 64 points on  $E_1 \times E_2 \times E_3$ , namely

$$z \in \left\{ \frac{1}{4} \begin{pmatrix} \pm\tau_1 \\ \pm(1+\tau_2) \\ \pm(1+\tau_3) \end{pmatrix} + \frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^3 \right\}.$$

Observe now that

$$\mathcal{L}_k \left( \frac{1}{4} + \frac{\tau_k}{4} \right)^2 = \mathcal{L}_k \left( \frac{1}{4} + \frac{\tau_k}{4} + \frac{1}{2} \right)^2 = -a_k,$$

whence  $\{\mathcal{L}_k(\frac{1}{4} + \frac{\tau_k}{4}), \mathcal{L}_k(\frac{1}{4} + \frac{\tau_k}{4} + \frac{1}{2})\} = \{\sqrt{-1}b_k, -\sqrt{-1}b_k\}$ . It is a straightforward computation to show that exactly 32 of them lie on  $\hat{X}_\nu$ .

□

**Lemma 4.3.** *Let  $S \rightarrow \hat{S} = \hat{X}/G$  be a regular generalized Burniat type surface with  $(\hat{X}, G) = (\hat{X}_\mu, G_2)$ . Let  $g \in \mathcal{G}_1$  be an element fixing point on  $X_\mu$ , then its fixed locus  $\text{Fix}(g)$  on  $\hat{X}_\mu$  is as in Table 4.*

	$Fix(g_i), X_\mu$
$g_1 := (0, 0, 0, 0, 0, 0, 1, 0, 0)$	4 genus 5 curves
$g_2 := (0, 0, 0, 1, 0, 0, 0, 0, 0)$	
$g_3 := (1, 0, 0, 0, 0, 0, 0, 0, 0)$	
$g_4 := (0, 0, 0, 1, 0, 0, 1, 0, 0)$	32 pt
$g_5 := (1, 0, 0, 0, 0, 0, 1, 0, 0)$	
$g_6 := (1, 0, 0, 1, 0, 0, 0, 0, 0)$	
$g_7 := (0, 0, 0, 0, 0, 1, 0, 0, 1)$	16 pt, 8 ell. curves
$g_8 := (0, 0, 1, 0, 0, 0, 0, 0, 1)$	
$g_9 := (0, 0, 1, 0, 0, 1, 0, 0, 0)$	
$g_{11} := (1, 0, 0, 0, 0, 1, 0, 0, 1)$	32 pt
$g_{12} := (0, 0, 1, 1, 0, 0, 0, 0, 1)$	
$g_{13} := (0, 0, 1, 0, 0, 1, 1, 0, 0)$	

TABLE 4.

*Proof.* Noting that  $g_0 = (1, 0, 0, 1, 0, 0, 1, 0, 0) \in G_2$ , the same arguments of the proof of Lemma 4.2 hold, and the statement follows.  $\square$

Finally, we consider the case  $\hat{X} = \hat{X}_b$ ; we study the fixed locus only of the elements in  $\mathcal{G}_0$  having fixed locus of dimension one on  $T$ , since it is enough for our purposes.

**Lemma 4.4.** *Let  $S \rightarrow \hat{S} = \hat{X}/G$  be a regular generalized Burniat type surface with  $(\hat{X}, G) = (\hat{X}_b, G_j)$ ,  $j \in \{3, 4\}$ . Let  $g \in \mathcal{G}_0$  be an element having fixed locus of dimension one on  $T$  then its fixed locus  $Fix(g)$  on  $\hat{X}_b$  is as in Table 5.*

	$Fix(g_i), X_b \text{ smooth}$	$Fix(g_i), X_b \text{ singular}$
$g_4 := (0, 0, 0, 1, 0, 0, 1, 0, 0)$	32 pt	16 pt, 8 nodes
$g_5 := (1, 0, 0, 0, 0, 0, 1, 0, 0)$		
$g_6 := (1, 0, 0, 1, 0, 0, 0, 0, 0)$		
$g_7 := (0, 0, 0, 0, 0, 1, 0, 0, 1)$	16 pt, 8 ell. curves	8 nodes, 8 ell. curves
$g_8 := (0, 0, 1, 0, 0, 0, 0, 0, 1)$		
$g_9 := (0, 0, 1, 0, 0, 1, 0, 0, 0)$		

TABLE 5.

*Proof.* By Proposition 3.11,  $\hat{X}_b$  is singular if and only if  $b \in B' = \{\pm 1, \pm a_i, \pm a_i a_j, \pm a_1 a_2 a_3\}$ , with  $i \neq j \in \{1, 2, 3\}$ ; in this case the eight nodes on  $\hat{X}_b$  are fixed by  $g_0 = (1, 0, 0, 1, 0, 0, 1, 0, 0)$ .

$g_4$ : it fixes the points  $(z_1, \overline{z_2}, \overline{z_3})$  with  $2\overline{z_2} = 2\overline{z_3} = 0$ . If  $b \notin B'$ , for every pair  $(\overline{z_2}, \overline{z_3})$ ,  $2z_1 \neq 0$ , whence  $g_4$  fixes 32 points on  $\hat{X}_b$ .

If  $b \in B'$ , for 8 choices of  $(\overline{z_2}, \overline{z_3})$  there are two values of  $z_1$  verifying the equation of  $\hat{X}_b$ , while for the other 8 possibilities there is a unique

value of  $z_1$  verifying the equation of  $\hat{X}_b$ , whence  $\text{Fix}(g_4)$  is given by 16 smooth points and the 8 nodes of  $\hat{X}_b$ .

$g_5$ - $g_6$ : these cases are analogous to  $g_4$ .

$g_7$ : since  $\text{Fix}(g_7) \cap \text{Fix}(g_0) = \emptyset$ , the fixed locus of  $g_7$  is independent from  $b$ .

The involution  $g_7$  fixes the points  $(z_1, \overline{z_2}, \overline{z_3})$  with  $2\overline{z_2} = \frac{1}{2}$ ,  $2\overline{z_3} = \frac{1}{2}$ . Let  $(\mathcal{L}_i(z_i)_0 : \mathcal{L}_i(z_i)_1)$  be the homogeneous coordinates of the point  $\mathcal{L}_i(z_i)$ . The equation of  $\hat{X}_b$  is then

$$\mathcal{L}_1(z_1)_0 \mathcal{L}_2(z_2)_0 \mathcal{L}_3(z_3)_0 = b_1 b_2 b_3 \mathcal{L}_1(z_1)_1 \mathcal{L}_2(z_2)_1 \mathcal{L}_3(z_3)_1.$$

It follows easily from the properties of the Legendre  $\mathcal{L}$ -function that  $(\mathcal{L}_i(\frac{1}{4})_0 : \mathcal{L}_i(\frac{1}{4})_1) = (0 : 1)$ ,  $(\mathcal{L}_i(\frac{1}{4} + \frac{\tau_i}{2})_0 : \mathcal{L}_i(\frac{1}{4} + \frac{\tau_i}{2})_1) = (1 : 0)$ . If  $z_2 = \pm\frac{1}{4}$  and  $z_3 = \pm\frac{1}{4} + \frac{\tau_3}{2}$  or  $z_2 = \pm\frac{1}{4} + \frac{\tau_2}{2}$  and  $z_3 = \pm\frac{1}{4}$ , then the equation is satisfied for any  $z_1 \in E_1$ , i.e  $g_7$  fixes 8 disjoint elliptic curves.

If  $z_2 = \pm\frac{1}{4}$  and  $z_3 = \pm\frac{1}{4}$  then the equation becomes  $\mathcal{L}_1(z_1)_1 = 0$  that has two solutions. If  $z_2 = \pm\frac{1}{4} + \frac{\tau_2}{2}$  and  $z_3 = \pm\frac{1}{4} + \frac{\tau_3}{2}$  then the equation becomes  $\mathcal{L}_1(z_1)_0 = 0$  that has two solutions, whence  $g_7$  fixes 16 isolated points and 8 disjoint elliptic curves on  $\hat{X}_b$ .

$g_8$ - $g_9$ : these cases are analogous to  $g_7$ .

□

We are now ready to prove our main result:

**Theorem 4.5.** *Let  $\epsilon: S \rightarrow \hat{S} = \hat{X}/G$  be a regular generalized Burniat type surface:  $(\hat{X}, G) \in \{(\hat{X}_\nu, G_1), (\hat{X}_\mu, G_2), (\hat{X}_b, G_j), j = 3, 4\}$ . Then it verifies the Bloch conjecture.*

We recall the following result, which allows to prove that each single surface in the moduli space corresponding to GBT surfaces of type  $j$  ( $1 \leq j \leq 4$ ) satisfies Bloch's conjecture.

**Theorem 4.6** ([BCF]).

- i) *Let  $S$  be a smooth projective surface homotopically equivalent to a GBT surface  $S_i$  of type  $i$ . Then  $S$  is a GBT surface of type  $i$ , i.e. contained in the same irreducible family as  $S_i$ .*
- ii) *The connected components  $\mathfrak{N}_i$  of the Gieseker moduli space  $\mathfrak{M}_{1,6}^{\text{can}}$  corresponding to GBT surfaces of type  $i$  is irreducible, generically smooth, normal and unirational of dimension 4 ( $i = 1, 2$ ) and of dimension 3 else.*

Together with Theorem 4.5 we thus obtain:

**Theorem 4.7.** *Let  $S$  be any surface such that its moduli point  $[S] \in \mathfrak{N}_i$ ,  $1 \leq i \leq 4$ , then  $S$  satisfies Bloch's conjecture.*

*Proof of Theorem 4.5.* We prove the statement case by case. In each case we consider a group  $H \cong (\mathbb{Z}/2\mathbb{Z})^2 < \text{Aut}(\hat{X})$  which allow us

define a group  $H' \cong (\mathbb{Z}/2\mathbb{Z})^2 < \text{Aut}(S)$  satisfying the assumptions of Corollary 1.5.

$G_1$ ) Let us consider the involutions  $\sigma_1 := (0, 0, 0, 0, 0, 0, 1, 0, 0)$  and  $\sigma_2 := (0, 0, 0, 1, 0, 0, 0, 0, 0)$  in  $\mathcal{G}'_1$ .

In the coset  $\sigma_1 G_1$  there are four elements fixing points on  $\hat{X}_\nu$ :

$$\begin{aligned} g_1 &= (0, 0, 0, 0, 0, 0, 1, 0, 0) & g_6 &= (1, 0, 0, 1, 0, 0, 0, 0, 0) \\ g_7 &= (0, 0, 0, 0, 0, 1, 0, 0, 1) & g_{15} &= (0, 1, 0, 1, 1, 1, 1, 1, 1) \end{aligned}$$

Since  $g_0 := (1, 0, 0, 1, 0, 0, 1, 0, 0) \in G_1$  has no fixed points on  $\hat{X}_\nu$ , the 4 sets  $\text{Fix}_{\hat{X}_\nu}(g_k)$  ( $k \in \{1, 6, 7, 15\}$ ) are pairwise disjoint.

$g_1$  fixes 4 curves of genus 5 on  $\hat{X}_\nu$ ,  $G_1$  acts transitively on this set of curves and  $g_0 := (1, 0, 0, 1, 0, 0, 1, 0, 0) \in G_1$  maps each curve onto itself, hence a genus 3 curve is fixed by  $\overline{\sigma_1}$  on  $\hat{S}$ .

$g_6$  and  $g_{15}$  fix 32 points each on  $\hat{X}_\nu$  and  $G_1$  acts freely on these two sets: we get 8 points fixed by  $\overline{\sigma_1}$ .

$g_7$  fixes 8 disjoint elliptic curves and 16 points if  $\hat{X}_\nu$  is smooth, 8 nodes otherwise. Since  $G_1$  acts freely on the set of points and transitively on the set of curves, we get that  $\overline{\sigma_1}$  fixes one elliptic curve and either 2 points or 1 node.

It follows that the involution  $\overline{\sigma_1}$  on  $\hat{S}$  lifts to an involution  $\sigma'_1$  on  $S$  whose fixed locus contains a genus 3 curve, an elliptic curve and 8 isolated smooth points, by Proposition 2.2,  $S/\sigma'_1$  is not of general type.

In the coset  $\sigma_2 G_1$  there are three elements fixing points on  $\hat{X}_\nu$ :

$$\begin{aligned} g_2 &= (0, 0, 0, 1, 0, 0, 0, 0, 0) & g_5 &= (1, 0, 0, 0, 0, 0, 1, 0, 0) \\ g_{11} &= (1, 0, 0, 0, 0, 1, 0, 0, 1) \end{aligned}$$

The 3 sets  $\text{Fix}_{\hat{X}_\nu}(g_k)$  ( $k \in \{2, 5, 11\}$ ) are pairwise disjoint, since  $g_0$  has no fixed points on  $\hat{X}_\nu$ .

$g_2$  fixes 4 curves of genus 5 on  $\hat{X}_\nu$ ,  $G_1$  acts transitively on this set of curves and  $g_0 := (1, 0, 0, 1, 0, 0, 1, 0, 0) \in G_1$  maps each curve onto itself, hence a genus 3 curve is fixed by  $\overline{\sigma_2}$  on  $\hat{S}$ .

$g_5$  fixes 32 points on  $\hat{X}_\nu$ : we get 4 points fixed by  $\overline{\sigma_2}$ .

If  $\hat{X}_\nu$  is smooth  $g_{11}$  fixes 32 isolated points, else it fixes 32 smooth isolated points and 8 nodes: we get that  $\overline{\sigma_2}$  fixes 4 smooth points and, if  $\hat{S}$  is singular, a node too.

It follows that the involution  $\overline{\sigma_2}$  on  $\hat{S}$  lifts to an involution  $\sigma'_2$  on  $S$  whose fixed locus contains a genus 3 curve, an elliptic curve and 8 isolated smooth points, by Proposition 2.2,  $S/\sigma'_2$  is not of general type.

Let  $\sigma_3 := \sigma_1 + \sigma_2$ . In the coset  $\sigma_3 G_1$  there are three elements fixing points on  $\hat{X}_\nu$ :

$$\begin{aligned} g_3 &= (1, 0, 0, 0, 0, 0, 0, 0, 0) & g_4 &= (0, 0, 0, 1, 0, 0, 1, 0, 0) \\ g_{14} &= (0, 1, 0, 0, 1, 0, 0, 1, 0) \end{aligned}$$



The 3 sets  $Fix_{\hat{X}_\nu}(g_k)$  ( $k \in \{3, 4, 14\}$ ) are pairwise disjoint, since  $g_0$  has no fixed points on  $\hat{X}_\nu$ .

$g_4$  and  $g_{14}$  fix 32 points each on  $\hat{X}_\nu$  and  $G_1$  acts freely on these two sets: we get 8 points fixed by  $\overline{\sigma_3}$ .

If  $\hat{X}_\nu$  is smooth,  $g_3$  fixes four disjoint smooth curves of genus 5 on  $\hat{X}_\nu$ . Let  $H := \langle (1, 0, 0, 1, 0, 0, 1, 0, 0), (0, 0, 0, 0, 0, 1, 1, 0, 1) \rangle \triangleleft G_1$ , each curve is invariant under  $H$ , hence  $\overline{\sigma_3}$  fixes two disjoint genus 2 curves.

If  $\hat{X}_\nu$  is singular,  $g_3$  fixes two disjoint smooth curves of genus 5 and 8 elliptic curves such that a point belongs to two of them if and only if it is a node. Looking at the  $G_1$  action on this configuration of curves, one can easily prove that  $\overline{\sigma_3}$  fixes on  $\hat{S}$  two elliptic curves intersecting in a node and a genus 2 curve.

It follows that the involution  $\overline{\sigma_3}$  on  $\hat{S}$  lifts to an involution  $\sigma'_3$  on  $S$  whose fixed locus contains a genus 2 curve and 8 isolated smooth points, by Proposition 2.2,  $S/\sigma'_3$  is not of general type.

Applying Corollary 1.5, with  $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \sigma_1, \sigma_2 \rangle$ , we conclude that  $S$  verifies Bloch's conjecture.

$G_2$ ) Let us consider the involutions  $\sigma_4 := (1, 0, 0, 0, 0, 0, 0, 0, 0)$  and  $\sigma_5 := (0, 0, 0, 1, 0, 0, 0, 0, 0)$  in  $\mathcal{G}_1$ .

In the coset  $\sigma_4 G_2$ , there are four elements fixing points on  $\hat{X}_\mu$ :

$$\begin{aligned} g_3 &= (1, 0, 0, 0, 0, 0, 0, 0, 0) & g_4 &= (0, 0, 0, 1, 0, 0, 1, 0, 0) \\ g_9 &= (0, 0, 1, 0, 0, 1, 0, 0, 0) & g_{12} &= (0, 0, 1, 1, 0, 0, 0, 0, 1) \end{aligned}$$

Since  $g_0 := (1, 0, 0, 1, 0, 0, 1, 0, 0) \in G_2$  has no fixed points, the 4 sets  $Fix_{\hat{X}_\mu}(l_j)$  ( $j \in \{3, 4, 9, 12\}$ ) are pairwise disjoint.

$g_3$  fixes four genus 5 curves that are invariant under the subgroup  $\langle (0, 0, 0, 1, 0, 1, 0, 0, 1), (1, 0, 0, 1, 0, 0, 1, 0, 0) \rangle \triangleleft G_2$ , hence  $\sigma'_4$  fixes two disjoint genus 2 curves.

$g_4$  and  $g_{12}$  fix 32 points each on  $\hat{X}_\nu$  and  $G_2$  acts freely on these two sets: we get 8 points fixed by  $\overline{\sigma_4}$ .

$g_9$  fixes 16 isolated points and 8 disjoint elliptic curves on  $\hat{X}_\mu$ , each curve is invariant under  $(1, 0, 1, 0, 0, 1, 0, 0, 0) \in G_2$ . We get that  $\sigma'_4$  fixes 2 isolated points and two disjoint elliptic curves.

It follows that  $Fix(\sigma'_4)$  is given by 10 isolated fixed points, two genus 2 curves and two elliptic curves. By Proposition 2.2, the quotient  $S/\sigma'_4$  is not of general type.

The same argument shows that  $S/\sigma'_4$  and  $S/(\sigma_4 + \sigma_5)'$  are not of general type, whence  $S$  verifies Bloch's conjecture, thanks to Corollary 1.5.

$G_3$ ) Let us consider the involutions  $\sigma_6 := (1, 0, 0, 1, 0, 0, 0, 0, 0)$  and  $\sigma_7 := (1, 0, 0, 0, 0, 0, 1, 0, 0)$ . In the coset  $\sigma_6 G_3$ , there are two elements

fixing points on  $\hat{X}$ :

$$g_6 = (1, 0, 0, 1, 0, 0, 0, 0, 0) \quad g_8 = (0, 0, 1, 0, 0, 0, 0, 0, 1)$$

The 2 sets  $\text{Fix}_{\hat{X}_b}(h_j)$  ( $j \in \{6, 8\}$ ) are disjoint.

$g_6$  fixes 32 points on  $\hat{X}_b$  if it is smooth, 16 smooth points and 8 nodes otherwise. Since  $G_3$  acts freely on this set of points we get that  $\overline{\sigma}_6$  fixes either 4 points ( $\hat{X}_b$  smooth) or 2 smooth points and the node.

$g_8$  fixes 16 isolated smooth points and 8 elliptic curves.

In the smooth case  $\text{Fix}(\sigma'_6)$  is given by 6 isolated fixed points and an elliptic curve. By Proposition 2.2, the quotient  $S/\sigma'_6$  is not of general type.

In the singular case,  $\overline{\sigma}_6$  fixes the node  $p$ , 4 smooth points and an elliptic curve. Let  $\Gamma := \epsilon^{-1}(p)$ , the involution  $\overline{\sigma}_6$  lifts to an involution  $\sigma'_6$  on  $S$  such that  $\sigma'_6(\Gamma) = \Gamma \cong \mathbb{P}^1$  and it fixes an elliptic curve, 4 smooth points and either two isolated points on  $\Gamma$  or  $\Gamma$ :  $(-2)$ -curve. In both cases, by Proposition 2.2,  $S/\sigma'_6$  is not of general type.

The same argument shows that  $S/\sigma'_7$  and  $S/(\sigma_6 + \sigma_7)'$  are not of general type, whence  $S$  verifies Bloch's conjecture, thanks to Corollary 1.5.

$G_4$ ) Considering involutions  $(1, 0, 0, 1, 0, 0, 0, 0, 0)$  and  $(1, 0, 0, 0, 0, 0, 1, 0, 0)$ , this case is analogous to the  $G_3$ -case.

□

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